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TUTTE POLYNOMIAL AND EHRHART POLYNOMIAL FOR ZONOHEDRON

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ABSTRACT

A polytope play a central role in different area of mathematics, for this we take of polytope which is known as a zonohedron then defined the matroid and arithmetic matroid. Multiplicity Tutte polynomial and Ehrhart polynomial to a zonohedron Z(X) in 2-dimension and 3-dimension are also given. A detailed for (D.Moci) theorem are proved by using multiplicity Tutte polynomial and establish some corollaries for the volume and the number of integral points of Z(X).

Theorem for the relation between the numbers of integral points on a zonohedron and the set of generating vectors with its proof is given. Combinatorial interpretation of the associated multiplicity Tutte polynomial with different examples is presented to demonstrate our results.

KEYWORDS: Ehrhart Polynomial, Tutte Polynomial, Zonohedron

INTRODUCTION

A zonohedron is a convex polyhedron where every face is a polygon with point symmetry or, equivalently symmetry under rotations through 180°. Any zonohedron may equivalently be described as the minkowski sum of a set of line segments in three –dimensional space.

Zonohedra were originally defined and studied by E.S. Fedorov, a Russian crystallographer. It is called zonotope because the faces parallel to each vector form so-called zone wrapping around the polytope P. In this paper, we focus on the case when P is a zonotope. Let X be a finite list of vectors in $\Lambda = \mathbb{Z}^n$. Assume that X spans $\mathbb{R}^n$ as a vector space, then

$$Z(X) \doteq \{ \ \sum_{x\in X} t_x x, \quad 0 \le t_x \le 1 \ \}.$$

is a convex polytope with integer vertices, called the zonotope of X. Zonotopes plays a crucial role in several areas of mathematics, such as hyperplane arrangements , box splines, and partition functions[7]. Several mathematical constructing from such a set X are: hyperplane arrangements and zonotopes in geometry, root systems and parking functions in combinatorices, for more information, Holts and Ron [10], introduced various algebraic structures containing a rich description of these objects.

Tutte polynomial is an invariant naturally associated to a matroid and encoding many of it is features which are the number of bases and their internal and external activity ([5], [6], [11]).A central role of this framework lie in the combinatorial notation of matroid, which axiomatizes the linear independence of the elements of X, where X is a finite list of vectors.

The present paper aims is to defined and investigate the Tutte polynomial $T_x(x, y)$ of matroid,

$$T_x(x, y) = \sum_{A\subseteq X} (x-1)^{n-r(A)} \ (y-1)^{|A|-r(A)}$$

Where, n means the dimension of lattice n-dimensional space $\mathbb{Z}^n$, r(A) is the rank of A, |A| is the cardinality of an independent subset of A.

The coefficients of the Tutte polynomial must be positive, then introduce the notation of arithmetic matroid (X, I, m) that is going to matroid (X,I) with multiplicity function m(A),A$\subseteq$ X which is notify in the next sections, The multiplicity matroid (X,I,m) and multiplicity Tutte polynomial $M_{X(x,y)}$ is the main subject of this paper. The relations with the zonotopes Z(X), a class of functions studied in approximation theory [9].

In addition, every arithmetic matroid associated an arithmetic(multiplicity) Tutte polynomial:

$$M_X(x,y) = \sum_{A \subseteq X} m(A) \ (x-1)^{n-r(A)} \ (y-1)^{|A|-r(A)}$$

where m(A) is the greatest common divisor will be present with more details in the next sections, this polynomial is defined in [4], it is shown to have several applications to vector partition functions, toric arrangements and zonotops. $M_X(x,y)$ Have also applications to graph theory, which have been described in [13].

The Ehrhart polynomial of a convex lattice polytope counts number of integer points in integral dilated of the polytope, this polynomial is a very important in many fields of mathematics, Therefore our first contribution is to establish a new relation satisfied by the coefficients of the Ehrhart polynomial.

In section three of this paper gives a method for finding the Ehrhart polynomial of the zonohedron Z(X),using the formula of the multiplicity(arithmetic) Tutte polynomial $M_X(x,y)$ such that:[1]

$$\mathcal{E}_x(q) = q^n \, M_x(1+\frac{1}{q},1)$$

Where, q means the dilated of the polytope.

PRELIMINARIES

This section is started by recalling the notations that we are going to introduce:

Definition (1): A matroid $\mathfrak{M}$ is a pair (X, I) where X is a finite set and I is a family of subsets of X (call the independent sets).Some properties of matroid:

- The empty set is independent.
- Every subset of an independent set is independent.
- Let A and B be two independent sets and assume that A has more elements than B. Then there is exist an element a $\in$ A \ B such that B $\cup\{a\}$ is still independent.

Example (1): X is a finite list of vectors of a vector space $\mathbb{R}^n$, independent=linearly independent.

Definition (2): A multiplicity matroid is the triple (X, I, m) where (X, I) is a matroid, m is a multiplicity function m: P(X) $\longrightarrow$ $N/\{0\}$, P(X) is a power set of X, [3]. We say that m is trivial multiplicity if it is identity equal to 1.

Definition (3): Let X$\subset \Lambda = Z^n$, for every A$\subseteq$ X, let r(A) is the rank of A, i.e. The number of all spanned subspace of $\mathbb{R}^n$[1].

The Tutte polynomial of the matroid is defined as, [7]:

$$T_x(x,y) = \sum_{A \subseteq X} \ (x-1)^{n-r(A)} \ (y-1)^{|A|-r(A)}$$

Where ,

n=the dimension of the lattice n-dimensional space Z^n.

|A|= the maximal cardinality of an independent subset of A.

Remark (1): A is independent↔ $r(A) = |A|$, where A⊆ X.

Definition (4): (X, I, m) is **representable**, means that the arithmetic(multiplicity) matroid is is realized by a list of elements in a finitely generated abelian group, the classical matroid (X, I) is said to be representable in characteristic 0 or (0-representable) if it is realized by a list of vectors in $\mathbb{R}^n$ [6].

Following [4], we denote $\langle A\rangle_{\mathbb{Z}}$ and $\langle A\rangle_{\mathbb{R}}$ respectively the sublattice of Λ and the subspace of $\mathbb{R}^n$ spanned by A. now define:

$$\Lambda_A \doteq \Lambda \cap \langle A\rangle_{\mathbb{R}}$$

The largest sublattice of Λ in which $\langle A\rangle_{\mathbb{Z}}$ has finite index. We defined m as this index, [1]:

$$m(A) \doteq [\Lambda_A : \langle A\rangle_{\mathbb{Z}}].$$

Notice that for every A⊂ X of a maximal rank, m(A) is equal to the greatest common divisor of the determinants of the basis extracted from A.

Definition (5): Let $X \subset \Lambda = Z^n$, for every A⊆ X, let r(A) is the rank of A, i.e. The number of all spanned subspace of $\mathbb{R}^n$[1].

The **(multiplicity) or arithmetic** Tutte polynomial of a multiplicity matroid

$$M_X(x, y) = \sum_{A\subseteq X} m(A)\ \ (x-1)^{n-r(A)}\ \ (y-1)^{|A|-r(A)}$$

Remark (2): The list X is unimodular if every basis B extracted from X spans Λ over $\mathbb{Z}$. (i.e. B has determinant=1) in this case $m(A) = 1$. for every A⊂ X then M_X(x, y) =T_X(x, y).

EHRHART POLYNOMIAL

Definition (6): let P⊂ $\mathbb{R}^d$be a lattice d-polytope, define a map L: N ⟶ N by

L (P, t) =card (tP∩ $\mathbb{Z}^n$),where 'card' means the cardinality of (tP∩ $\mathbb{Z}^n$)and N

Is the set of natural numbers and tp is the dilated polytope. It is seen that L (P, t) can be represented as: L (P, t) $=\sum_{i=1}^{d} c_i\, t^i$, this polynomial is said to be the Ehrhart Polynomial of a lattice d-polytope P, [16].

Remark (3): let P⊂ $\mathbb{R}^d$be a lattice 2-polytope, the Ehrhart polynomial of P is given

By:

$$L(P, t) = At^2 + \frac{1}{2}Bt + 1$$

Where A is the area of the polytope and B is the number of lattice points on the boundary of P,[16].

Ehrhart Polynomial with Multiplicity Tutte Polynomial: [1]

The secondary result in this paper is:

$$\mathcal{E}_x(q) = q^n \, M_x(1 + \frac{1}{q}, 1)$$

Ehrhart dilated multiplicity Tutte polynomial

Polynomial

Now some theorems are given below with it's proof:

THEOREMS

Definition (7): let P⊂ $\mathbb{R}^d$ be a lattice d-polytope. For t ∈ $\mathbb{Z}^+$,the set

tP = {tX :X∈ P} is said to be the dilated polytope.

In the next proofs we use $qX \doteq \{qx, x \in X\}$. as the dilated polytope, the same meaning of above definition just change the variables.

Proposition (1): [1]

Let m(qA) be the multiplicity function of the dilated list then:

$$m(qA) = q^{r(A)} m(A)$$

Lemma (1):[1]

Let M_{qX} (x, y) be the multiplicity Tutte polynomial of the dilated polytope qX, then

$$M_{qX}\ (x, y) = q^n M_X\left(\frac{x-1}{q} + 1, y\right)$$

Proof

By defined:

$$M_{qX}(x, y) = \sum_{A \subseteq X} m(qA)\ \ (x-1)^{n-r(A)}\ (y-1)^{|A|-r(A)}$$

Since,

$$m(qA) = q^{r(A)} m(A)$$

Then,

$$M_{qX}(x, y) = \sum_{A \subseteq X} q^{r(A)} m(qA)\ \ \ (x-1)^{n-r(A)}\ (y-1)^{|A|-r(A)}$$

Divided above equation by $\frac{q^n}{q^n}$ we get,

$$M_{qX}(x, y) = \sum_{A \subseteq X} q^n\ m(A)\ \frac{(x-1)^{n-r(A)}}{q^{n-r(A)}}\ .\ (y-1)^{|A|-r(A)}.$$

$$= \sum_{A \subseteq X} q^n\ m(A)\ \left(\frac{x-1}{q}\right)^{n-r(A)}\ \ (y-1)^{|A|-r(A)}$$

Now we added 1 and subtract 1 from $\left(\frac{x-1}{q}\right)$ it is yeilds that:

$$= \sum_{A \subseteq X} q^n \, m(A) \left(\frac{x-1}{q} - 1 + 1\right)^{n-r(A)} (y-1)^{|A|-r(A)}$$

Therefore

$$M_{qX}(x,y) = q^n M_X\left(\frac{x-1}{q} + 1, y\right).$$

■

Theorem (1): (D. Moci), [2]

Let v ∈X and set $X_1 = X_2 := X\{v\}$, If rk ({v}) =1 and rk (X\ {v}) =rk(X), then

$$M_X(x,y) = M_{X_1}(x,y) + M_{X_2}(x,y)$$

Before proof theorem (1) we introduce two fundamental constructions**, (deletion and contraction):** which are natural reductions for many network models arising from awide range of problems at the hearts of computer science, engineering, optimization, physics, and biology.

Definition (8): let (X, I, m) be an arithmetic matroid, $v \in X$ and set $X_1 = X_2 = X\backslash\{v\}$, then the triple $(X_1, I, m_1)$is **the deletion** of v, i.e. $rk_1(A) = rk(A)$ and $m_1(A) := m(A)$ for all $A \subseteq X_1$.

Definition (9): let (X, I, m) be an arithmetic matroid, $v \in X$ and set $X_1 = X_2 = X\backslash\{v\}$, then the triple$(X_2, I, m_2)$ be **the contraction** of v, i.e.

$$rk_2(A) := rk(A \cup \{v\}) - rk(\{v\}) \text{ and } m_2(A) := m(A \cup \{v\}) \text{ for all } A \subseteq X_2.$$

Once, can proof theorem (1) immediately

Proof

The sum expressing $M_X(x,y)$ splits into two parts. The first is over the sets

$$A \subseteq X_1,$$

$$M_{X_1}(x,y) = \sum_{A \subseteq X_1} m(A) \, (x-1)^{r(X)-r(A)} \, (y-1)^{|A|-r(A)}$$

r(X) =rk(X) is the rank of X.

Since clearly r(X) = r (X_1). The second part is over the sets A, $\lambda \in A$,where λ is a non-zero elements. For such sets we have that:

$$|\overline{A}| = |A| - 1, r(\overline{A}) = r(A) - 1, r(X_2) = r(X) - 1, m(A) = m(\overline{A}).$$

Therefore

$$M_{X_2}(x,y) = \sum_{A \subseteq X, \lambda \in A} m(A) \, (x-1)^{r(X)-r(A)} \, (y-1)^{|A|-r(A)} = \sum_{\overline{A} \subseteq X_2} m(\overline{A}) \, (x-1)^{r(X_2)-r(\overline{A})} \, (y-1)^{|\overline{A}|-r(\overline{A})}. \blacksquare$$

Corollary (1): [1]

The number $|Z(X) \cap \Lambda|$ of integer points in the zonotope is equal to $M_X(2,1)$.

Proof: by applying deletion-contraction. We can reduce to the case in which X is a basis of U, U is the real vector space.

Such that $U = A \otimes \mathbb{R}$ then in this basis Z(X) is parallelpieped.

For every face F we define A_F as subset of X corresponding to the coordinates which are not constant of F. Since all the other coordinates are identically equal either to 0 or to 1, for every $A \subseteq X$ there are exactly 2^k faces F s.t. $A_F = A, k = |X \backslash A|$ among these faces the only contributing to $M_X(1,1)$. Is the one whose Constant coordinates are all equal to 0 i.e. Z (A). On the other hand, to compute the total number of integer points we have to take all these 2^k faces .since any two of them are disjoint and contain the same number of points. In their interior by $M_X(x,1) = \sum_{k=0}^{n} |J_k(x)| X^k$ see [4]

Corollary (2): [1]

The volume (Z(X)) of the zonotope is equal to $M_X(1,1)$.

Proof

Z(X) is paved by a family of polytopes $\{\prod_B\}$, where B varies among all the Bases extracted from X.And every $\prod_B$ is obtained by translating the zonotope Z(B) generated by the list B.

Hence

Vol. $(\prod_B) = |\, det(B)|$

However, when B is a basis

$m(B) = [\Lambda : \langle B \rangle z\,] = |\, det\ (B)\,|$

Since

$$M_X(1,1) = \sum_{B \subset X, B\ basis} m(B).$$

The claim is follows.

Now, according to the above definitions and theorem we get the following theorem with its proof:

Theorem (2)

Let $X \subset \Lambda = Z^n$, the number of integral points for the zonohedron equals to the multiplicity (m(qA)) iff the set of vectors X is a unit vectors and the number of components is n.

That is $\mathcal{E}_x(q) = \sum_{A \subseteq X} m(qA)$ iff X is unit vectors.

Proof

If $\mathcal{E}_x(q) = \sum_{A \subseteq X} m(qA)$, let r(A) is the rank of A, i.e. The number of all spanned subspace of $\mathbb{R}^n$.

Then $\mathcal{E}_x(q) = q^{r(A)} \sum_{A \subseteq X} m(A)$

Since the number of components is n, then r(A)=n= the number of cardinality.

Then |A|=r(A)

Only holds if X is a unit vectors.

Conversely,

If |A|=r(A) then

$\mathcal{E}_x(q) = q^n \, M_x(1 + \frac{1}{q}, 1)$

$= q^n \sum_{A \subseteq X} m(A) \quad (x-1)^{n-r(A)} \quad (y-1)^{|A|-r(A)}$

$= q^n \sum_{A \subseteq X} m(A)$

$$\mathcal{E}_x(q) = \sum_{A \subseteq X} m(qA),$$

Some examples that described the methods of compute the multiplicity Tutte polynomial and associated Ehrhart polynomial are given:

Example (1)

In this example let X= {(1, 1), (1,-1)} $\subseteq Z^2$, with dimension n=2,

X can be written as

$X=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

According to the formula given below,

$$M_X(x, y) = \sum_{A \subseteq X} m(A) \quad (x-1)^{n-r(A)} \quad (y-1)^{|A|-r(A)}$$

$m(A)$ need to compute for every $A \subseteq X$,

$m(\phi)=1, m(\{v_1\}) = 1 = m(\{v_2\}),$

$m(\{v_1, v_2\}) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = |-2| = 2$

Put the obtained result in the formula we get

$M_X(x, y) = x^2 - 2x + 1 + 2(x-1) + 2 = x^2 + 1.$

Example (2)

In this example, let X be = {(3, 3), (1,-1), (2, 0)} $\subseteq \mathbb{Z}^n$ with dimension n=2, then X can be written as

$X = \begin{pmatrix} 3 & 1 & 2 \\ 3 & -1 & 0 \end{pmatrix}$

According to the formula given below,

$M_X(x, y) = \sum_{A \subseteq X} m(A) \, (x-1)^{n-r(A)} \, (y-1)^{|A|-r(A)}$

After some computation, we get

$m(\phi) = 1$,$m(v_1)=3$,$m(v_2)=1$,$m(v_3)=2$

$m(\{v_1, v_2\})=2$,$m(\{v_1, v_3\})=6$,$m(\{v_2, v_3\})=2$

$m(\{v_1, v_2, v_3\}) = 2.$

Put the obtained results in the formula above we get,

$M_x(x, y) = x^2 + 4x + 2y + 7$

Then put the result above in the Ehrhart polynomial we get,

$\mathcal{E}_x(q) = q^n \, M_x(1 + \frac{1}{q}, 1)$

Where, q means the dilation of the polytope.

$\mathcal{E}_x(q) = 14q^2 + 6q + 1.$

q	1	2	3	4	5	6
Number of integral point	21	69	145	249	381	541

Example (3)

In this example consider the list in $\mathbb{Z}^2$

X = {(3, 0), (0, 2), (1, 1)} with dimension n=2,

Then X can be written as

$X=\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

According to the formula given below,

$M_X(x, y) = \sum_{A \subseteq X} m(A) \; (x-1)^{n-r(A)} \; (y-1)^{|A|-r(A)}$

After some computation, we get

$m(\phi) = 1$,$m(v_1)$=3 ,$m(v_2)$=2 ,$m(v_3)$=1

$m(\{v_1, v_2\})$=6 ,$m(\{v_1, v_3\})$=3 ,$m(\{v_2, v_3\})$=2

$m(\{v_1, v_2, v_3\})$ =1

Put the obtained results in the formula above we get,

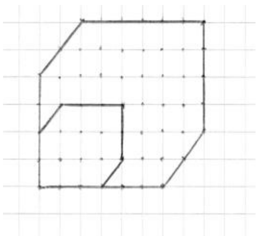
$M_X(x, y) = (x-1)^2 + (3+2+1)(x-1) + (6+3+2) + (y-1) = x^2+4x+y+5$

Then put the result above in the Ehrhart polynomial we get,

$\mathcal{E}_x(q) = q^n \, M_x(1 + \frac{1}{q}, 1)$

Where, q means the dilation of the polytope.

$\mathcal{E}_x(q) = 11q^2 + 6q + 1$



q	1	2	3	4	5	6
Number of integral point	18	57	118	201	306	433

Example (4)

In this example let X= {(1,0, 0), (0,1,0), (0,0,1)} $\subseteq \mathbb{Z}^3$ with dimension n=3, then X can be written as

X=$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, X is the set of generating vectors of unit cube.

According to the formula given below,

$$M_X(x,y) = \sum_{A \subseteq X} m(A)\ (x-1)^{n-r(A)}\ (y-1)^{|A|-r(A)}$$

After some computation, we get

$m(\phi) = 1$,$m(v_1)$= $m(v_2)$= $m(v_3)$=1

$m(\{v_1, v_2\})$=1 ,$m(\{v_1, v_3\})$=1 ,$m(\{v_2, v_3\})$=1

m $(\{v_1, v_2, v_3\})$ =1,

Put the obtained results in the formula we get,

$M_x(x,y)$=$x^3 + 1$.

Put the result above in the Ehrhart polynomial we get,

$$\mathcal{E}_x(q) = q^n\ M_x(1 + \frac{1}{q}, 1)$$

Where q means the dilation of the polytope.

$$\mathcal{E}_x(q) = q^3 + 3q^2 + 3q + 1.$$

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